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Instantons, compactification and S-duality in $\mathcal{N} = 4$ SUSY Yang-Mills theory II

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Abstract: We present a semiclassical calculation of instanton effects in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory formulated on $\mathbb{R}^3 \times S^1$ and also in the $\mathcal{N} = 1$ theory obtained by introducing chiral multiplet masses. In the $\mathcal{N} = 4$ case, these instanton effects are related to the bulk contribution to the index which counts BPS dyons in the corresponding four dimensional theory. In both cases, the calculations provide semiclassical tests of recently proposed exact results for the lowest non-trivial terms in the derivative expansion of the wilsonian effective action.

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Instantons provide a rare example of a non-perturbative effect in quantum field theory that can be analysed quantitatively. In this paper, which is a sequel to [1], we investigate instanton effects in four dimensional supersymmetric gauge theories compactified to three dimensions on a circle of circumference β. In particular, we consider $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with gauge group SU(2), as well as the so-called $\mathcal{N}=1^*$ theory obtained by introducing $\mathcal{N}=1$-preserving mass terms. When formulated on $\mathbb{R}^3\times S^1$ both these theories have finite action instantons which correspond to BPS saturated magnetic monopoles in four dimensions. Various other field configurations also contribute as saddle points of path integral in the compactified theory. In both these theories, it is possible to obtain exact information about the contributions of these instantons to the low-energy effective action on the Coulomb branch. In fact, by exploiting the constraints of supersymmetry, together with some additional asumptions about the behaviour of the theory at strong-coupling one may compute the first non-trivial term in the derivative expansion of the effective action exactly. In the $\mathcal{N}=4$ case, the relevant term is the eight fermion coupling calculated in [1] (see also [2]–[6] for related work in the three-dimensional limit). In the $\mathcal{N}=1^*$ case, we consider instead two-fermion terms which are determined by the exact superpotential derived in [7]. In both these cases we will compare the predictions of these exact results with the results of direct semi-classical calculations. The comparison yields a detailed agreement between these two approaches. As we review below, the $\mathcal{N}=4$ and $\mathcal{N}=1^*$ results are closely related and the corresponding instanton calculations are very similar. In the following we will discuss the $\mathcal{N}=4$ case in detail and then sketch the corresponding story in the $\mathcal{N}=1^*$ theory.

In the $\mathcal{N}=4$ case, we find an interesting connection between instanton effects in the compactified theory and the existence of BPS saturated bound states in the corresponding four-dimensional model. As emphasized by Sen [8], the SL$(2,\mathbb{Z})$ duality of the $\mathcal{N}=4$ theory in four dimensions implies the existence of BPS saturated states with electric and magnetic charges q and k, whenever these two integers are coprime. The existence of these states can be investigated in the weak coupling limit using the moduli space approximation [9]. The moduli space of k BPS monopoles is a smooth hyper-Kähler manifold with an isometric decomposition,

$$\mathcal{M}_k = R^3 \times \frac{S^1\times\tilde{\mathcal{M}}_k}{Z_k}\,. \tag{1}$$

Sen showed that the existence of the required bound states is equivalent to the existence of certain normalizable supersymmetric groundstates of quantum mechanics on the reduced moduli space $\tilde{\mathcal{M}}_k$. In particular, for each k, there should be exactly one groundstate with charge p under the Z_k symmetry appearing in the denominator of (1) whenever p is coprime to k. By standard arguments the wavefunction of each of these groundstates is an L^2 normalizable middle-dimensional harmonic form on $\tilde{\mathcal{M}}_k$. The existence of such a form can be checked explicitly in the case $k = 2$ using the hyper-Kähler metric on $\tilde{\mathcal{M}}_2$ given by Atiyah and Hitchin [10]. As the metric is unknown for $k > 2$, there has so far been no direct verification of Sen's prediction in these cases, although lower bounds on the dimension of the L^2 cohomology of $\mathcal{M}_k$ can be proven using only the topology of this manifold [11].

A standard approach to exhibiting the existence of supersymmetric ground states is to compute the Witten index $\mathrm{Tr}(-1)^F \exp(-\beta H)$. In the present case the Witten index is precisely the index of the laplacian acting on forms on $\tilde{\mathcal{M}}_k$. As we are dealing with quantum mechanics on a non-compact manifold, an important caveat is that we should only count normalizable harmonic forms. This necessitates defining an appropriate L^2 index. In this case we adopt the approach used in [12, 13] to study the related problem of checking the existence of the normalizable threshold boundstates of D0-branes required by IIA/M duality. Let $\mathcal{I}_{L^2}(p,k)$ denote the L^2 index of the laplacian on $\tilde{\mathcal{M}}_k$ restricted to states with definite Z_k charge p. As usual, one contribution to the index is an integral of a suitable index density over the manifold. In the present case, this is essentially just the Gauss-Bonnet integral of the Euler density over the monopole moduli space. The main result of our semi-classical calculation is that this bulk contribution arises as a coefficient of a term in the instanton expansion of the compactified theory.[1] A prediction for the bulk contribution can therefore be extracted from the exact results of [1]. In particular we find that the bulk contribution to $\mathcal{I}_{L^2}(p,k)$ is equal to unity for each value of p and k. However the index also recieves contribution from a boundary or defect term. This term and the resulting application to the problem of counting BPS boundstates in four dimensions is discussed in detail in [14] and, for completeness, we will briefly summarize the relevant argument below. Taking into account both the boundary and bulk contributions, the resulting index is in accord with Sen's prediction. Finally, we consider the instanton contributions to the $\mathcal{N} = 1^*$ superpotential which also turns out to be proportional to the Gauss-Bonnet integral over the corresponding monopole moduli space.

We begin by considering the $\mathcal{N} = 4$ theory with gauge group SU(2) in four dimensions. The theory has a Coulomb branch on which the gauge symmetry is broken to U(1). The massless bosonic fields on the Coulomb branch include the abelian gauge field of the unbroken U(1) as well as six scalar fields ϕ_a with $a = 1, \ldots 6$ tranforming in the vector representation of the R-symmetry group, $Spin(6)_{\mathcal{R}} = \mathrm{SU}(4)_{\mathcal{R}}$. The low energy theory also contains left and right handed Weyl fermions transforming in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $\mathrm{SU}(4)_{\mathcal{R}}$. In terms of $\mathcal{N} = 1$ supermultiplets, the $\mathcal{N} = 4$ theory contains a gauge multiplets and three chiral multiplets in the adjoint representation of the gauge group. The latter can be represented in $\mathcal{N} = 1$ superspace

[1] A possible connection between instanton effects in three dimensions and the Sen boundstate problem in four dimensions was suggested in [4]

as chiral superfields Φ_i, with $i = 1, 2, 3$. The $\mathcal{N} = 1^*$ theory is obtained by introducing $\mathcal{N} = 1$ supersymmetric mass terms for each chiral superfield. The resulting superpotential is,

$$W = \frac{1}{g^2}\mathrm{Tr}\left(\Phi_1[\Phi_2, \Phi_3] + m_1\Phi_1^2 + m_2\Phi_2^2 + m_3\Phi_3^2\right). \tag{2}$$

The fields are normalized so that the coupling constant appears only as an overall prefactor of the complete action. In four dimensions, these mass terms lift the Coulomb branch of the $\mathcal{N} = 4$ theory leaving isolated vacua.

At the classical level, the massless fields of the $\mathcal{N} = 1^*$ theory are those of the minimal supersymmetric SU(2) gauge theory in four dimensions: the gauge field and a single Weyl fermion in the adjoint representation of the gauge group. After compactification, the component of the gauge field in the compact direction gives rise to a Wilson line, ω. When the Wilson line acquires a non-zero VEV, the gauge group is broken down to U(1) and the theory is in a Coulomb phase. Thus the compactified $\mathcal{N} = 1^*$ theory has a Coulomb branch parametrized by ω. The Wilson line is shifted by integer multiples of 2π by large gauge transformations: $\omega \to \omega + 2n\pi$. We will refer to the integer n as the index of the corresponding gauge transformation. After identifying gauge equivalent configurations, ω becomes a periodic variable with period 2π.

In three dimensions a massless abelian gauge field is dual to a scalar σ. This field enters initially as a Lagrange multiplier for the Bianch identity and therefore gives rise to a surface term $S_\sigma = ik\sigma$ in the action where,

$$k = \frac{1}{4\pi}\int d^3x \vec{\nabla}\cdot\vec{B}, \tag{3}$$

where $B_i = \varepsilon_{ijk}F^{jk}/2$ is the abelian magnetic field of the low energy theory and k is the corresponding magnetic charge. Because k is quantized in integer units, the dual photon σ, like the Wilson line ω, is effectively a periodic variable of period 2π. After a duality transformation we obtain the classical effective action for the massless bosonic fields [15],

$$S_B = \frac{2g^2}{\beta(8\pi)^2}\int d^3x \left[\left(\frac{4\pi}{g^2}\partial_\mu\omega\right)^2 + \left(\partial_\mu\sigma + \frac{\theta}{2\pi}\partial_\mu\omega\right)^2\right]. \tag{4}$$

The low energy theory also includes a single Weyl fermion which is neutral under the U(1) gauge group of the low energy theory. The classical Coulomb branch of the $\mathcal{N} = 1^*$ theory is therefore T^2/Z_2 where T^2 is the two dimensional torus parametrized by ω and σ and the Z_2 quotient corresponds to the Weyl group of SU(2). The Coulomb branch has a natural complex structure with holomorphic coordinate $Z = -i(\tau\omega + \sigma)$. Thus we have a complex torus E with complex structure parameter τ. In this context, S-duality simply corresponds to invariance under modular transformations of the

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complex torus E [16]. To make $\mathcal{N}=1$ SUSY manifest, we can promote scalar Z to a chiral superfield denoted by the same letter. The lagrangian for ω and σ can then be written in $\mathcal{N}=1$ superspace as,

$$\mathcal{L}_{\rm eff} = \int d^2\theta d^2\bar{\theta}\, \mathcal{K}_{cl}[Z,\bar{Z}] \tag{5}$$

with the classical Kähler potential $\mathcal{K}_{cl} = Z\bar{Z}/(8\beta\,{\rm Im}\tau)$.

As discussed above, the $\mathcal{N}=4$ theory already has a Coulomb branch in four dimensions parametrized by the expectation values of the six adjoint scalars of the theory. After compactification and a three dimensional duality transformation the Coulomb branch is parametrized by ϕ_a, ω and σ and is a copy of $(\mathbb{R}^6 \times T^2)/\mathbb{Z}_2$. As above the $\mathbb{Z}_2$ quotient corresponds to the Weyl group of SU(2). Although the unbroken R-symmetry of the lagrangian is just the $Spin(6)_{\mathcal{R}}$ of the four-dimensional theory, the resulting supersymmetry algebra with sixteen supercharges in three dimensions has a larger $Spin(8)_{\mathcal{R}}$ group of automorphisms. To make this manifest, we combine the eight scalar fields in a vector with components X_l, with $l=1,\ldots,8$ as:

$$\vec{X} = \left(\sqrt{\delta}\phi_1, \ldots, \sqrt{\delta}\phi_6, \sqrt{\gamma}\omega, \sqrt{\epsilon}\left(\sigma + \frac{\theta\omega}{2\pi}\right)\right) \tag{6}$$

with $\delta = \beta/g^2$, $\gamma = 1/g^2\beta$ and $\epsilon = g^2/(16\pi^2\beta)$. The enlarged R-symmetry is such that $\vec{X}$ transforms in the eight-dimensional vector representation $\mathbf{8}_V$ of $Spin(8)_{\mathcal{R}}$. In the following Roman indices $l,m,n,o = 1,\ldots,8$ label the components of this repesentation. The bosonic part of the classical effective action is $S^B_{\rm eff} + ik\sigma$, where k is the integer valued magnetic charge and,

$$S^B_{\rm eff} = \int d^3x\, \frac{1}{2}\, \delta_{lm}\partial_\mu X^l \partial^\mu X^m\,. \tag{7}$$

This is the action of a three dimensional non-linear σ-model whose target manifold is $\mathcal{M}_{cl} = (\mathbb{R}^6 \times T^2)/\mathbb{Z}_2$ with the standard flat metric. The low energy theory also includes eight Majorana fermions ψ^Ω_α, with $\Omega = 1,\ldots,8$ and $\alpha = 1,2$, which comprise one of the two eight-dimensional spinor representations of $Spin(8)_{\mathcal{R}}$ denoted $\mathbf{8}_S$. Note that the $Spin(8)_{\mathcal{R}}$ symmetry is broken to $Spin(6)_{\mathcal{R}}$ solely by the periodic boundary conditions on the bosonic fields X_7 and X_8;

$$\begin{aligned} X_7 &\sim X_7 - 2n_2\Omega_2 \\ X_8 &\sim X_8 - 2n_1\Omega_1 - 2\kappa n_2\Omega_2 \end{aligned} \tag{8}$$

with $2\Omega_1 = g/2\sqrt{\beta}$, $2\Omega_2 = 2\pi/g\sqrt{\beta}$ and $\kappa = \theta g^2/8\pi^2$.

As discussed in [1], the compactified $\mathcal{N}=4$ theory can be realized on the world volume of two D-branes in type-II string theory on $\mathbb{R}^9 \times S^1$. There are two equivalent D-brane descriptions related to each other by T-duality. We can either consider two

D3-branes of the IIB theory wrapped on S^1 or two unwrapped D2-branes of the IIA theory on the dual circle. Separation of the branes in their six common non-compact transverse directions corresponds to turning on the six scalar fields ϕ_a. In the IIA picture, the distance between the branes in the compact transverse direction corresponds to the Wilson line. The VEV of the dual photon corresponds to the extra compact transverse dimension which appears after lifting the IIA brane configuration to M-theory [17].

We will begin by analysing possible instanton corrections to the low energy effective action. In the weak coupling limit, the path integral is dominated by field configurations of minimum action in each topological sector. Gauge field configurations in the compactified theory are labelled by two distinct kinds of topological charge [18]. The first is the Pontryagin number carried by instantons in four-dimensions,

$$p = \frac{1}{8\pi^2} \int_{R^3 \times S^1} \mathrm{Tr}\,[F \wedge F]\,. \tag{9}$$

The second is the magnetic charge k, defined in (3) above. The 4D instanton number appears in the microscopic action as the term $-ip\theta$. For each value of p, the action is minimized by configurations which solve the (anti-)self-dual Yang-Mills equation. An important feature of the compactified theory which differs from the theory on R^4 is that p is not necessarily quantized in integer units.

For fixed non-zero magnetic charge, the minimum action configurations satisfy a Bogomol'nyi equation which depends explicitly on the choice of vacuum on the Coulomb branch. It is useful to combine the non-zero component of the scalar field with the three-dimensional gauge field A_i to form a four dimensional gauge field. The relevant Bogomol'nyi equation is simply a self-dual Yang-Mills equation in this four dimensional theory. We will begin by discussing the special case where Higgs fields ϕ_a vanish for $a = 1, \ldots 6$, and only the periodic fields ω and σ are non-zero. For brevity we will call this case 1. The special feature of case 1 is that the four dimensional theory in question can be identified with the four dimensional $\mathcal{N} = 4$ theory we started with! The other extreme case (which we will call case 2) is when ϕ_a are non-zero and the Wilson line vanishes. In this case we can use the $Spin(6)_{\mathcal{R}}$ symmetry to rotate the Higgs field so that only one component, say ϕ_1, is non-zero. One may then construct an auxiliary four dimensional gauge field $v_{\tilde{\mu}}$ with $\tilde{\mu} = 1, 2, 3, 4$ where $v_{\tilde{\mu}} = A_i$ for $\tilde{\mu} = i = 1, 2, 3$ and $v_4 = \phi_1$. It is also useful to define an auxiliary $\mathcal{N} = 4$ supersymmetric Yang-Mills theory based on our new gauge field $v_{\tilde{\mu}}$. As explained in [1], this construction can be understood by recaling that the four-dimensional $\mathcal{N} = 4$ theory can be derived by dimensional reduction of $\mathcal{N} = 1$ SUSY Yang-Mills in ten dimensions. If the original theory is obtained by dimensional reduction of the $x_4, \ldots, x_9$ directions, the auxiliary theory corresponds instead to reduction of x_0 and $x_5, \ldots, x_9$. Static BPS solutions which depend only on x_i with $i = 1, 2, 3$, can also be thought of as static self-dual configurations in the auxiliary theory. The advantage of this viewpoint is that, if we define chirality with respect to the four dimensions of the auxiliary theory, the BPS monopole is invariant under the eight right-handed supercharges.

In case 1, the relevant instantons are described in detail in [7, section 2] and we will now briefly recall these results. The basic configurations which carry magnetic charge are the standard BPS monopole solutions of the Bogomol'nyi equation which we will refer to as 3D instantons. For magnetic charge k, these have action with real part $|k|\omega(4\pi/g^2)$. In addition to their magnetic charge, these instantons also carry fractional Pontryagin number $p = k\omega/2\pi$. Including surface terms the complex action is $S_k = -i\tau\omega|k| + ik\sigma$. The action can be concisely written in terms of the complex scalar field $Z = -i(\tau\omega + \sigma)$ as $S_k = kZ$ for $k > 0$ and $S_k = k\bar{Z}$ for $k < 0$. When $|\phi| = 0$, the four-dimensional theory also has periodic instanton solutions of integer Pontryagin number p and euclidean action $-2\pi ip\tau$ for $p > 0$ and $2\pi ip\bar{\tau}$ for $p < 0$. These solutions persist after compactification to three dimensions. When the scale size of the instanton is much smaller than radius of compactification, the instanton field configuration of the compactified theory is close to that of the four dimensional model. Hence we will refer to these configurations as 4D instantons. However, when the scale size approaches the radius of compactification the relevant field configurations are quite different from their counterparts in the four dimensional theory. These configurations are also known as calarons.

As explained in [7], there are many additional instantons which must also be taken into account. In particular, it is notable that the k instanton contribution, which is proportional to $\exp(-kZ)$ respects the periodicity in σ but is not a periodic function of the Wilson line. In fact, under a shift $\omega \to \omega + 2\pi$, we have $\exp(-kZ) \to q^k \exp(-kZ)$ where $q = \exp(2\pi i\tau)$ is the factor associated with a four dimensional instanton. This simply reflects the fact that the BPS monopole is not invariant under the large gauge transformation which leads to this 2π shift in the Wilson line [19]. As above, such gauge transformations are classfied by an index n which is an element of $\pi_1(S^1) = Z$. In fact performing a large gauge transformation with index n leads to two infinite towers of configurations for each value of the magnetic charge. For $k, n > 0$, we find configurations with magnetic charge $+k$ and Pontryagin number $p = kn + k\omega/2\pi$ which yield semiclassical contributions proportional to $q^{kn} \exp(-kZ)$. For $n < 0$, the sign of the magnetic charge is flipped yielding a contribution proportional to $q^{kn} \exp(+kZ)$. Note that, despite having negative magnetic charge, these configurations are *not* anti-monopoles: they are self-dual rather than anti-self-dual. In fact, for anti-self dual configurations, the same same formulae apply but with Z replaced by $\bar{Z}$ (and τ by $\bar{\tau}$).

The configurations described above are straightforward to understand in the context of the IIA brane picture. As before we have two D2 branes on $\mathbb{R}^9 \times S^1$ where the compact dimension is transverse to the brane world volume. The branes are seperated on S^1 by an amount proportional to the Wilson line ω. The IIA theory contains D0-branes with mass $4\pi/g^2$ (in units with $\sqrt{\alpha'} = 0$) to unity. The duality between the IIA string theory and M-theory, requires that any number of D0-branes form a bound state at threshold. These bound states are identified with the Kaluza-Klein modes of the eleven dimensional metric and the number of constituent D0-brane corresponds to momentum in the M-direction. In the present context, the VEV of the dual photon is naturally interepreted as the spatial coordinate in the eleventh direction. Noting that magnetic monopoles of charge k contribute to the path integral with a phase $\exp(ik\sigma)$ we immediately see that magnetic charge should be identified with D0-brane number. Each of the instantons of magnetic charge $\pm k$ described in preceeding paragraphs corresponds to a configuration where the euclidean worldline of a boundstate of k D0-branes is stretched between the two D2-branes in the compact dimension. From this picture it is clear that there are two basic 3D instantons for each positive value of k corresponding to the two topologically inequivalent paths (of length less than 2π) between the two branes. Paths which wind around the S^1 yield an infinite tower of instantons over each of these basic configurations. The winding number of the path is precisely $|n|$, where n is the index of the large gauge transformations introduced above. A three dimensional limit in field theory corresponds to decompactifying the S^1 direction in the IIA picture while keeping the distance between the D2-branes fixed. Clearly only the configurations with winding number $n = 0$ contributes in this limit. In the present case with $|\phi| = 0$, taking a four-dimensional limit leads to the conformal point of the $\mathcal{N} = 4$ theory where there are no monopole solutions.

According to our discussion above the total Pontryagin number of the configurations specified by integers k and n, should be equal to kn. Comparing this with the IIA brane suggests that Pontryagin number or 4d instanton number should be identified by the total winding number of the constituent D0-branes. This fact can be understood via T-duality in the compact dimension. In the IIB picture introduced above, a four dimensional Yang-Mills instanton appears as a D-instanton on the D3-brane worldvolume. After T-duality each D-instanton becomes a single D0-brane with euclidean worldline wrapped around the S^1. This picture also reveals that a single 4D instanton in the compactified theory contains two 3D instantons of opposite magnetic charges as constituents [19]. Each of the configurations with $k > 0$ described above (as well as each of the 4D instantons with $p > 0$) obeys the four-dimensional self-dual Yang-Mills equation. Hence each is invariant under half the generators of the supersymmetry algebra. The remaining generators act non-trivially on the background fields and generate exactly eight fermion zero modes. These generators and the resulting zero modes each have the same four-dimensional chirality. Although the Callias index theorem indicates that the 3D instantons should have many additional zero modes, these are lifted by the Yukawa couplings and other related terms in the lagrangian [3, 7].

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Our discussion has so far been limited to case 1 described above where the VEVs of the four-dimensional scalar fields vanish. In the $\mathcal{N}=4$ theory, this describes only the two dimensional submanifold of the Coulomb branch. However, in the $\mathcal{N}=1^*$ theory, mass terms for the Higgs fields lift the additional directions in the Coulomb branch, leaving the two-dimensional complex torus parametrized by the complex scalar $Z=-i(\tau\omega+\sigma)$. In other words, case 1 is all we need to analyse the instanton corrections on the Coulomb branch of the $\mathcal{N}=1^*$ theory. The mass terms also lift six out of the eight exact fermion zero modes the magnetic instantons then contribute terms proportional to $\exp(-kZ)$ [20] to a holomorphic superpotential, $\mathcal{W}$, for the $\mathcal{N}=1$ chiral superfield with lowest component Z. The superpotential is uniquely determined by holomorphy, double-periodicity and the known Witten index of the theory in question. The main result of [7] was that $\mathcal{W}(Z)=m_1m_2m_3\mathcal{P}(Z)$, where $\mathcal{P}(Z)$ is the Weierstrass function. The semiclassical content of the $\mathcal{N}=1^*$ result can be understood from the standard expansion of the Weierstrass function for $\tau\to\infty$,

$$\begin{aligned}\mathcal{W} = m_1m_2m_3 &\sum_{k=1}^{\infty} k\exp(-kZ) + \\ &+m_1m_2m_3\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} kq^{kn}\left[\exp(-kZ)+\exp(+kZ)-2\right]. \end{aligned} \tag{10}$$

Each of the instantons described above contributes to this sum. Note that the coefficient of each instanton term is independent of the coupling indicating that there are no perturbative corrections to the superpotential in the instanton background. However we will find below that the semiclassical definition of the field Z appearing in the exact superpotential is modified by perturbative effects.

Returning to the $\mathcal{N}=4$ theory, we consider a generic point on the Coulomb branch with VEVs for the six scalar fields of the four-dimensional theory turned on. This leads to several modifications which we now discuss. Firstly the four-dimensional Higgs fields contribute to the action of each magnetic instanton. As above we obtain an infinite tower of states by acting with large gauge transformations. An examination of the Bogomol'nyi equation reveals that, after a large gauge transformation of index n, the action of a BPS monopole with magnetic charge k becomes, $k(4\pi/g^2)\sqrt{\beta^2|\phi|^2+|\omega-2n\pi|^2}$ This formula is obvious in the IIA brane picture described above. Turning on the VEV $|\phi|$ corresponds to moving the two D2-branes apart in one of their six transverse directions. The first term under the square root simply reflects the consequent increase in the length of the euclidean worldline of a boundstate of k D0-branes which stretches between the D2-branes and also winds around the compact direction n times.

There is also an important modification to the structure of the fermion zero modes when VEVs for the four dimensional Higgs fields are introduced. As discussed above we can always reformulate the Bogomol'nyi equation as a self-duality condition for a four dimensional gauge field. However, the four-dimensional theory in question no longer corresponds to the original four dimensional theory on $R^3 \times S^1$. Hence, the corresponding magnetic instanton has eight exact fermion zero modes of single four-dimensional chirality with respect to the auxiliary four-dimensional theory discussed above. This leads to an eight-fermion term in the Coulomb branch effective action of the $\mathcal{N} = 4$ theory of the form,

$$\mathcal{L}_{8f} = \mathcal{V}(\sigma, \omega, |\phi|) \prod_{A=1}^{4} \bar{\rho}^{A}_{\dot{\delta}} \bar{\rho}^{A\,\dot{\delta}}\,, \tag{11}$$

where $\bar{\rho}^{A}_{\dot{\delta}}$, with $A = 1, \ldots, 4$ and $\dot{\delta} = \dot{1}, \dot{2}$, are the right-handed Weyl fermions of the auxiliary theory. These are linear combinations of the Weyl fermions λ_α and $\bar{\lambda}_{\dot{\alpha}}$ of the original four dimensional $\mathcal{N} = 4$ theory. Similarly, the index $A = 1, \ldots, 4$ is a spinor index of a $Spin(6)$ subgroup of the $Spin(8)_{\mathcal{R}}$ of the three dimensional SUSY algebra. This subgroup coincides with the (broken) $Spin(6)$ R-symmetry group of the auxiliary theory rather than the unbroken $Spin(6)_{\mathcal{R}}$ of the compactified theory.

The main result of [1] was an exact formula for the eight fermion term (11). The result is very closely related to the exact superpotential for the $\mathcal{N} = 1^*$ theory. As in that case, the function $\mathcal{V}(\sigma, \omega, |\phi|)$ must be invariant under periodic shifts of the Wilson line ω and of the dual photon σ. Supersymmetry also demands that $\mathcal{V}$ is a harmonic function on the Coulomb branch. This restriction is analogous to the holomorphy of the superpotential in the $\mathcal{N} = 1^*$ case. Finally, given these restrictions, the required function can then be determined uniquely from its behaviour at singularities in both cases.

The resulting formula for $\mathcal{V}(\sigma, \omega, |\phi|)$ ([1, equation (16)]) can be expanded in two different regimes. The two expansions both correspond a series of semiclassical instanton corrections coming from configurations with non-zero magnetic charge. The relevant instantons contributing to the two expansions were identified in [1] as wrapped branes in the IIA and IIB D-brane pictures respectively. The IIA series, which is valid $g^2 << 1$, has precisely the form anticipated above: it involves a double summation over two integers k and n, to be identified with the magnetic charge and with the topological index of a large gauge transformation respectively. Each term has an exponential supression proportional to,

$$\exp\left(-\frac{4\pi k}{g^2}\sqrt{\beta^2|\phi|^2 + |\omega - 2\pi n|^2} + ik\left(\sigma + n\theta + \frac{\theta\omega}{2\pi}\right)\right). \tag{12}$$

The exponent agrees with the instanton action described above including the contribution of the surface terms. The formulae of [1] also predict the exact coefficient with which each of these terms contribute.

On the other hand, for $\beta|\phi| \gg 1$, we have an alternative series for $\mathcal{V}$ which we will refer to as the IIB series. This has the form of a double sum over integers k and q

identified as magnetic and electric charges respectively. Each term has characteristic exponential supression,

$$\exp\left(-\beta M(q,k) - iq\omega + ik\sigma\right) \tag{13}$$

with an explicit prediction for the coefficient. Here $M(q,k)$ is precisely the mass of a BPS saturated state of the four dimensional theory with electric charge q and magnetic charge k,

$$M(q,k) = |\phi|\sqrt{k^2\left(\frac{4\pi}{g^2}\right)^2 + \left(q + k\frac{\theta}{2\pi}\right)^2}\,. \tag{14}$$

Note that the periodic θ dependence of the IIA instanton expansion has been resummed to reproduce the characteristic shift in the electric charge of a BPS dyon discovered by Witten in [29]. The series admits an interpretation as a trace over the BPS sector of the Hilbert space of the $\mathcal{N}=4$ theory in four dimensions. which schematically of the form, $\mathrm{Tr}_{BPS}\exp(-\beta H + i\sigma K - i\omega Q)$ where H is the four dimensional hamiltonian and K and Q are magnetic and electric charge operators respectively. The coefficient of the eight fermion vertex essentially an index which counts only BPS saturated states just as the Witten index counts only supersymmetric ground states. Similar refinements of the Witten index have been considered before in the context of two dimensional field theories with four supercharges [21] and also appear in string theory [22]. In the following we will use this interpretation in the semiclassical limit to relate the coefficients appearing in the IIB series to the bulk contributions to the L^2 index theory which counts the monopole and dyon boundstates required for S-duality of the four dimensional theory.

To make direct contact with semiclassical calculations, we still need to impliment the weak-coupling limit. This is most evident in the IIB expansion where the BPS dyon mass appearing in the exponent has a non-trivial expansion in g^2. The explicit prediction for the leading non-trivial behaviour in the $g^2 \to 0$ limit is,

$$\mathcal{V}_{IIB} = \sum_{k=1}^{\infty}\sum_{q=-\infty}^{+\infty}\tilde{\mathcal{V}}_{q,k}\exp(ik\sigma - iq\omega) \tag{15}$$

with

$$\tilde{\mathcal{V}}_{q,k} = \left(\frac{\beta}{g^2}\right)^8 \beta\,\frac{k^{11/2}}{(\beta M)^{5/2}}\exp\left(-\beta k M - \frac{\beta q^2}{2k\Lambda}\right). \tag{16}$$

Where $M = M(0,1) = (4\pi/g^2)|\phi|$ is the mass of a single BPS monopole and $\Lambda = M/|\phi|^2$.

As explained in [1], there is also a complimentary semiclassical limit for the IIA series. The explicit formula for $\mathcal{V}$ is,

$$\mathcal{V}_{IIA} = \sum_{k=1}^{\infty}\sum_{n=-\infty}^{+\infty}\mathcal{V}_{k,n}\exp ik\left(\sigma + n\theta + \frac{\theta\omega}{2\pi}\right) \tag{17}$$

with

$$\mathcal{V}_{k,n} = \left(\frac{\beta}{g^2}\right)^9 \frac{k^6}{(\beta M)^3} \exp\left(-\beta k M - \frac{1}{2}k\Lambda\frac{(\omega-2\pi n)^2}{\beta}\right). \tag{18}$$

In fact these two series are exactly equal: $\mathcal{V}_{IIA} = \mathcal{V}_{IIB}$. The equality is a consequence of the Poisson resummation formula and corresponds to the well known semiclassical exactness of the path integral for a non-relativistic particle moving on a circle [23]. The circle in question is the global part of the unbroken U(1) gauge group.

We will now perform a direct semiclassical calculation of the instanton contributions to the eight fermion vertex. For simplicity, we will set $\theta = 0$ and work in the case 2 setup with $\omega = 0$. The vertex yields a non-zero contribution to the large distance behaviour of the eight fermion correlation function,

$$\mathcal{G}^{(8)}(x_1,\ldots,x_8) = \langle \prod_{A=1}^{4} \rho_1^A(x_{2A})\rho_2^A(x_{2A-1})\rangle \tag{19}$$

and we will compare this contribution to the result of a semiclassical calculation of the same quantity.

As usual the path integral reduces to an integral over the instanton collective coordinates in the semiclassical limit. Thus we begin by considering the moduli space $\mathcal{M}_k$ of static charge k monopole solutions of the four dimensional theory. This is a hyperkähler manifold of real dimension $4k$ with a standard isometric decomposition,

$$\mathcal{M}_k = R^3 \times \frac{S^1 \times \tilde{\mathcal{M}}_k}{Z_k}. \tag{20}$$

The R^3 factor of moduli space corresponds to the position, $\vec{R}$, of the monopole in three-dimensional space. The angular variable $\chi \in [0, 2\pi]$ which parametrizes the S^1 factor corresponds to the action of global U(1) gauge transformations on the monopole. The remaining collective coordinates of the charge k solution, denoted Y_q with $q = 1, \ldots, d = 4(k-1)$, are coordinates on the reduced moduli space $\tilde{\mathcal{M}}_k$. As indicated in (20), there is a Z_k symmetry, acting both on χ and on the Y_q, which relates gauge equivalent configurations and must be divided out.

The monopole also has a total of $8k$ linearly independent fermion zero modes which give rise to Grassmann collective coordinates. Of these eight are generated by the action of half the sixteen supersymmetry generators. These modes are all of the same chirality with respect to the four dimensions of the auxiliary theory and we denote the corresponding eight Grassmann collective coordinates, ξ_δ^A with $A = 1, \ldots 4$ and $\delta = 1, 2$. These parameters appear explicitly in the long-range behaviour of the fermionic fields ρ_δ^A, defined in the previous section, in the monopole background. Specifically we have $\rho_\delta^A \equiv \rho_\delta^{cl\,A}$ for $|\vec{r} - \vec{R}| \gg |\phi|^{-1}$ with [3],

$$\rho_\delta^{cl\,A} = 8\pi k \xi_\gamma^A S_F(\vec{r} - \vec{R})_\delta^\gamma, \tag{21}$$

where $S_F(\vec{r}) = \tau_i r^i/4\pi|\vec{r}|^3$ is the free fermion propagator in three dimensions. The Grassmann parameters corresponding to the additional zero modes are denoted α^q_δ with $q = 1, \ldots 4(k-1)$ and $\delta = 1, 2$. These are the superpartners of the Y_q. In the semiclassical limit we replace ρ by ρ^{cl} in (19) and the path integral reduces to an integral over the instanton moduli space. In a conventional instanton calculation the collective coordinates are c-numbers and we simply integrate over them. In the present context, we are working on $\mathbb{R}^3 \times S^1$ and we must also allow for field configurations which depend periodically on the euclidean time coordinate $\tau = x_0$. This 'time dependence' then leads to a residual path integration over the collective coordinates.

In order to analyse the time dependent contributions it is useful to start from the four dimensional theory in Minkowski space, where the monopoles are BPS states in the Hilbert space. In four dimensions the semiclassical dynamics of monopoles should be described by supersymmetric quantum mechanics on the moduli space [9, 24, 25]. This moduli-space approximation is justified because the monopole is very massive at weak coupling, and the times derivatives of the fields are small. The effective lagrangian of moduli space quantum mechanics splits up as,

$$L_{QM} = L_R + L_\xi + L_\chi + L^{(k)}_{\mathrm{Rel}} \,. \tag{22}$$

Where the first three terms are free lagrangians for $\vec{R}$, ξ and χ respectively. In particular we have $L_R = kM|\dot{\vec{R}}|^2/2$ which is the lagrangian for a non-relativistic particle of mass $M(0,k) = kM$ moving on R^3 where $M = M(0,1) = (4\pi/g^2)|\phi|$ is the mass of a single monopole. Similarly we have $L_\chi = k\Lambda\dot{\chi}^2/2$ which is the lagrangian for a particle of mass $k\Lambda$ moving on a circle parametrized by $\chi \in [0, 2\pi]$. Here $\Lambda = M/|\phi|^2$ is the moment of inertia of a single monopole with respect to global gauge rotations. These two terms are supersymmetrized by a free lagrangian L_ξ for the fermionic degrees on freedom ξ^A_δ. The last term, L_{Rel} describes the dynamics of the relative degrees of freedom of the k monopole solution. In particular we have [24, 25],

$$L^{(k)}_{\mathrm{Rel}} = \frac{1}{2}\left[g_{pq}\, d_\tau Y^p d_\tau Y^q + g_{pq}\, i\bar{\alpha}^p\gamma^0 D_\tau\alpha^q + \frac{1}{12}R_{pqrs}(\bar{\alpha}^p\alpha^r)(\bar{\alpha}^q\alpha^s)\right], \tag{23}$$

where $\bar{\alpha} = \alpha\gamma^0$ with $\gamma^0 = \sigma^2$, and $D_\tau\alpha^q = d_\tau\alpha^q + d_\tau Y^r\Gamma^p_{rq}\alpha^q$ is the covariant derivative on $\tilde{\mathcal{M}}_k$ formed from the hyper-Kähler metric g_{pq}. The resulting lagrangian (23) includes kinetic terms for Y_q and their superpartners α^q_δ as well as a four fermion term which couples to the Riemann curvature tensor R_{pqrs} on the moduli space. The lagrangian describes quantum mechanical non-linear σ-model with eight supercharges having the hyper-Kähler manifold $\tilde{\mathcal{M}}_k$ as the target space.

In order to describe instanton effects on $\mathbb{R}^3 \times S^1$, we will simply continue the effective quantum mechanics of the collective coordinates to a compact euclidean time dimension. Thus we have,

$$\mathcal{G}^{(8)}(x_1, x_2, \ldots, x_8) = \int [d^3R(\tau)]\,[d\chi(\tau)]\,[d^8\xi(\tau)]\,[d^{4k-4}Y(\tau)]\,[d^{8k-8}\alpha(\tau)] \times \\ \times \prod_{A=1}^{4} \rho_1^{cl\,A}(x_{2A})\rho_2^{cl\,A}(x_{2A+1}) \,\exp\left(\int_0^\beta d\tau\, L_{QM}\right). \tag{24}$$

The path integral measure is defined by the usual Feynman-Kac prescription starting from the canonical quantization of the system described by the lagrangian (23). Indeed the usual correspondence to canonical quantization means that the above expression has a simple interpretation in terms as a trace over the Hilbert space which we will explore in the following. However, we will now focus on the direct evaluation of the quantum mechanical path integral.

To begin with we notice that (24) factorizes into two seperate parts. In particular, as the field insertions depend only of the translational coordinates $\vec{R}$ and their $\mathcal{N}=4$ superpartners ξ_α^A, we have $\mathcal{G}^{(8)} = \mathcal{G}^{(8)}_{\rm COM} \times \mathcal{Z}_k$ with

$$\mathcal{G}^{(8)}_{\rm COM}(x_1, x_2, \ldots, x_8) = \int [d^3R(\tau)]\,[d^8\xi(\tau)] \prod_{A=1}^{4} \rho_1^{cl\,A}(x_{2A})\rho_2^{cl\,A}(x_{2A+1}) \times \\ \times \exp\left(-\int_0^\beta d\tau\, L_R + L_\xi\right) \tag{25}$$

and,

$$\mathcal{Z}_k = \int [d\chi(\tau)]\,[d^{4k-4}Y(\tau)]\,[d^{8k-8}\alpha(\tau)] \,\exp\left(-\int_0^\beta d\tau\, L_\chi + L^{(k)}_{\rm Rel}\right). \tag{26}$$

Both path integrations on S^1 are performed with periodic boundary conditions for all fields. Notice that, because of the Z_k quotient appearing in (20), the COM charge angle χ does not decouple completely from the remaining degrees of freedom. For this reason it has been grouped together with the relative degrees of freedom rather than the centre of mass variables.

As the center of mass degrees of freedom are free, evaluating (25) is very straightforward. In particular we may evaluate this path integral in the semiclassical approximation which is in any case exact for a free system. In fact, because of the periodic boundary conditions the only paths which contribute are the trivial ones $\vec{R}(\tau) = \vec{R}(0)$ and $\xi_\delta^{(A)}(\tau) = \xi_\delta^{(A)}(0)$ and the path integral reduces to an ordinary integral. In other words the COM part of the calculation is essentially independent of the compactification scale β (apart from an overall power of β which is fixed by dimensional analysis). The Grassmann integrals corresponding to the eight exact fermion zero modes of the instanton are saturated by the eight explicit field insertions. The resulting integral is a convolution of eight three-dimensional propagators which can be interpreted as an eight fermion vertex in the effective action,

$$\mathcal{L}_{8f} = \mathcal{V}(|\phi|, \omega, \sigma) \prod_{A=1}^{4} \bar{\rho}^{A}_{\dot{\delta}} \bar{\rho}^{A\,\dot{\delta}} \tag{27}$$

with $\mathcal{V} = \sum_{k=1}^{\infty} \mathcal{W}_k$ and,

$$\mathcal{W}_k = \left(\frac{\beta}{g^2}\right)^8 \beta \, \frac{k^{11/2}}{(\beta M)^{5/2}} \, \mathcal{Z}_k \, \exp\left(-k\beta M + ik\sigma\right). \tag{28}$$

All that remains is to evaluate $\mathcal{Z}_k$. We begin by considering the limit $\beta \to 0$, where the analysis coincides with the discussion of the three dimensional theory given in [3]. In this case each of the path integrations in (26) reduced to an ordinary integral and the partition function factorizes as $\mathcal{Z}_k = \mathcal{Z}_k^{\chi} \mathcal{Z}_k^{\rm Rel}$ with,

$$\mathcal{Z}_k^{\chi} = \int_0^{\frac{2\pi}{k}} d\chi \sqrt{\frac{2\beta k}{g^2\phi}} = \frac{2\pi}{k}\sqrt{\frac{2\beta k}{g^2\phi}} \tag{29}$$

$$\mathcal{Z}_k^{\rm Rel} = \frac{1}{(8\pi)^{d/2}(d/2)!} \int \frac{\prod_{q=1}^{d} dY^q}{\sqrt{\det(g)}} \, \varepsilon^{p_1 p_2 \ldots p_d} \, \varepsilon^{q_1 q_2 \ldots q_d} \, R_{p_1 p_2 q_1 q_2} \ldots R_{p_{d-1} p_d q_{d-1} q_d}\,, \tag{30}$$

where the range of integration over χ in (29) reflects the Z_k quotient in (20). We recognize the right hand side of (30) as the bulk contribution to the Witten index of supersymmetric quantum mechanics on $\tilde{\mathcal{M}}_k$. By the Gauss-Bonnet (GB) theorem this integral is formally equal to the Euler character of this manifold. However, because $\tilde{\mathcal{M}}_k$ is non-compact, the GB integral may differ from the Euler character by a boundary term. Comparison with the exact formula (18), reveals that our result (29), (30) implies that $\mathcal{Z}_k^{\rm Rel} = k$. Interestingly Segal and Selby [11] have shown that the Euler character of $\tilde{\mathcal{M}}_k$ is also equal to k. This is consistent with our result provided that the boundary contribution to the GB theorem on $\tilde{\mathcal{M}}_k$ vanishes. In the case $k = 2$ [26], one may check explicitly that this is the case using the exact metric of Atiyah and Hitchin. For $k > 2$, the corresponding metric is unknown and it is not possible to check directly that the boundary terms vanish. However, the three dimensional analysis of [4] and [3] shows that the prediction that the GB integral on $\tilde{\mathcal{M}}_k$ is equal to k can be derived (for all k) assuming only supersymmetry and the absence singularities away from the origin of the Coulomb branch.

In this approach we can also consider the contribution of instantons with non-zero winding number n. Consider a monopole of magnetic charge k. The charge angle χ may vary slowly with euclidean time τ as determined by the lagrangian $L_\chi = k\Lambda\dot{\chi}^2/2$. The periodic boundary conditions require $\chi(\beta) = \chi(0)$ so the admissable classical paths are $\chi(\tau) = 2\pi n\tau/\beta$. The classical action of such a path is precisely $2k\Lambda\pi^2 n^2/\beta$. Adding in the action of the static monopole $\beta k M$ we reproduce the exponent of (18) in the case $\omega = 0$. Thus we are summing over periodic classical dyon-like solutions with angular velocity $\dot{\chi} = 2\pi n/\beta$ which orbit the compact dimension n times. Finally note that because the collective coordinate χ parametrises global U(1) gauge transformations. As $\tau = x_0$, the x_0-dependence is just that of a large gauge transformation of winding number n acting on the monopole in harmony with our earlier description of these configurations.

As discussed in [1] and above, the path integral expression for the correlation function $\mathcal{G}^{(8)}$ has a natural interpretation in the hamiltonian formalism:

$$\begin{aligned}\mathcal{G}^{(8)}(x_1, \ldots, x_8) &= \left\langle \prod_{A=1}^{4} \rho_1^A(x_{2A})\rho_2^A(x_{2A-1}) \right\rangle \\ &= \mathrm{Tr}\left[(-1)^F \prod_{A=1}^{4} \rho_1^A(x_{2A})\rho_2^A(x_{2A-1}) \exp\left(-\beta H - i\omega Q + i\sigma K\right)\right], \quad (31)\end{aligned}$$

where H is the four-dimensional hamiltonian and Q and K are the electric and magnetic charge operators respectively (which have integer eigenvalues q and k). The BPS states discussed above then contribute to the trace (31) with the exponential supression $\exp(-\beta M(q,k) - iq\omega + ik\sigma)$. The fact that non-BPS configurations have additional zero modes means that they do not contribute to the correlation function in question. The corresponding statement in the hamiltonian formulation is that the eight fermionic insertions in (31) effectively act as a projection operator onto the BPS sector of the Hilbert space and that *only* BPS states contribute to the trace.

It is also useful to reconsider the semiclassical calculation of $\mathcal{Z}_k$ in the hamiltonian framework. In particular we have,

$$\mathcal{Z}_k = \mathrm{Tr}\left[(-1)^F \mathrm{P}_k \exp\left(H_\chi + H^{(k)}_{\mathrm{Rel}} + i\sigma K - i\omega Q\right)\right], \tag{32}$$

where H_χ and H_{rel} are the collective coordinate hamiltonian obtained by Legendre transform from the lagrangian terms L_χ and L_{rel} respectively. Here P_k is a projector, to be defined explicitly below, which impliments the Z_k quotient in (20). The hamiltonian for χ is simply $H_\chi = 1/2k\Lambda\dot{\chi}^2$ where $\Lambda = M/|\phi|^2$. This is the hamiltonian for a free particle of mass $k\Lambda$ moving on S^1. The conjugate momentum to χ is the electric charge $Q = k\Lambda\dot{\chi}$ and the eigenstates of H_χ are states integer electric charge q and energy $q^2/2k\Lambda$. The Hilbert space of SUSY quantum mechanics on $\tilde{\mathcal{M}}_k$ splits up into states of definite Z_k charge $p = 0, \ldots k-1$ [8] and we define seperate Witten indices for each sector as,

$$\mathcal{I}(p,k) = \mathrm{Tr}_p\left[(-1)^F \exp\left(-\beta H^{(k)}_{\mathrm{Rel}}\right)\right], \tag{33}$$

where Tr_p denotes restriction of the trace is evaluated only on states of Z_k charge p. As explained by Sen [8], the precise effect of the Z_k quotient in (20) is to impose a selection rule on the electric and Z_k charges q and p. Specifically we are instructed only to retain states with $p + q = 0 \bmod k$. Setting $q = -p + ks$ for integer s, we find that,

$$\mathcal{Z}_k = \sum_{p=0}^{k-1} \sum_{s=-\infty}^{+\infty} \mathcal{I}(p,k) \exp\left(-\frac{\beta(-p+ks)^2}{2k\Lambda} + i(-p+ks)\omega\right). \tag{34}$$

The final semiclassical result for the total instanton contribution to the eight fermion vertex coefficient is $\mathcal{V} = \sum_{k=1}^{\infty} \mathcal{W}_k \exp(-k\beta M + ik\sigma)$ with $\mathcal{W}_k$ given by (28) and (34). We find that this agrees precisely with the prediction for $\mathcal{V}_{IIA} = \mathcal{V}_{IIB}$ as given in (15), (16) if and only if[2]

$$\mathcal{I}(p,k) = +1 \tag{35}$$

for each magnetic charge $k > 0$ and for each Z_k charge $p = 0, 1, \dots k-1$.

If we interpret $\mathcal{I}(p,k)$ as the true Witten index which counts all normalizable harmonic forms of Z_k charge p on $\tilde{\mathcal{M}}_k$ then our result is quite unexpected. It suggests that there is (at least) one normalizable middle-dimensional harmonic form for each value of p and k. If p and k are coprime this confirms the presence of the forms required by Sen's conjecture. However it also implies the existence of harmonic forms corresponding to unwanted boundstates in the non-coprime cases. In fact this point needs to be considered more carefully because of non-compactness. The appropriate context is the L^2 index theory developed for the purpose of counting D0-brane boundstates in [12, 13]. In the non-compact case a regulated Witten index must be defined in finite volume. The index depends explicitly on the dimensionless parameter $\mu = \beta/R$ where R is the length scale corresponding to the regulator. The bulk contribution to the index is obtained by taking the limit $\beta \to 0$ with R held fixed. However as the answer can only depend on the ratio μ this is equivalent to removing the regulator while keeping β fixed. For this reason we identify the quantity $\mathcal{I}(p,k)$ appearing in our semiclassical calculation as the bulk contribution to the L^2 index $\mathcal{I}_{L^2}(p,k)$ and our main result is that this quantity is equal to unity for each value of k and p.

In fact it is also possible to analyse the boundary contribution to the L^2 index after making certain simplfying assumptions first suggested (in the context of D0-brane bound-states) by Yi [13]. This is discussed in detail in [14] and we will now briefly summarize the main points. We begin by considering the case of k identical BPS monopoles with zero electric charge. As the monopole moduli space is smooth, the boundary contribution to the L^2-index is determined solely by the asymptotic region of large seperation. However, the moduli-space metric is asymtotically flat relecting the fall-off of the velocity-dependent forces between monopoles at large distance. This suggests that the boundary contribution to the L^2-index is should be identical to that of an appropriate free problem where the interactions between

[2] This equality also requires a precise value for the overall, p and k independent, normalization constant. However this was fixed in the three-dimensional limit in [3].

monopoles are neglected. In this approximation, the monopole moduli space reduces to the symmetric product of k copies of $\mathbb{R}^3 \times S^1$ and the relevant partition function is that of the corresponding supersymmetric orbifold quantum mechanics. The absence of bound-states in this free approximation suggests that the analysis in this case is trivial. However, the vanishing of the L^2 index in the free case actually arises as a cancellation between non-zero boundary and bulk contributions. In particular one may evalute the bulk contribution by an explicit evaluation of the relevant path integral which is given in [14]. After taking into account the number of Bose and Fermi degrees of freedom (which differs from the D0-brane case originally considered by Yi) we find the bulk contribution to the L^2 index of the free problem is one and therefore that the desired boundary term is minus one. However, we have argued above that the bulk contribution for the full quantum mechanics on the monopole moduli space is also one. Hence the final answer for the L^2 index in this case is zero, indicating the absence of a threshold boundstate of k magnetic monopoles in the absence of electric charge. This is consistent with Sen's prediction.

If the electric and magnetic charges contain a common factor l, the relevant moduli space contains an asymptotic region describing l far-seperated clusters each of charges $(p/l, k/l)$. To analyse the boundary contribution in this case we may pursue an inductive approach assuming the validity of Sen's conjecture for all $k' < k$. In this case, the asymptotic region of the moduli space described above can be identified with the configuration space of l indistinguishable particles of charges $(p/l, q/l)$ which can be accurately described by an appropriate quantum mechanical sigma model with target the orbifold $\mathrm{Sym}^l(\mathbb{R}^3 \times S^1)$. As above this yields a boundary contribution of minus one and gives a result of zero in accord with Sen's conjecture for magnetic charge k. Finally, in the case of p and k coprime, there are no asymptotic regions of the moduli space corresponding to identical particles and the boundary contribution vanishes. The resulting value of the L^2 index is one which confirms the presence of the boundstates states predicted by Sen in this case. We refer the reader to [14, section 3] for further details.

In the remainder of the paper we will present a related calculation which provides a non-trivial semiclassical test of the exact superpotential $\mathcal{W} = m_1 m_2 m_3 \mathcal{P}(Z)$ of the $\mathcal{N} = 1^*$ theory. In the $\mathcal{N} = 4$ case, each BPS instanton had exactly eight unlifted fermion zero modes, two for each of the four species of Weyl fermions present in the theory. The $\mathcal{N} = 1^*$ theory differs from $\mathcal{N} = 4$ SYM only by the inclusion of mass terms for the adjoint scalars and their $\mathcal{N} = 1$ superpartners. In particular this includes masses m_1, m_2 and m_3 for three out of the four species of Weyl fermions. The main effect of these masses is to lift six out of the eight exact fermion zero modes described above. The instantons with two remaining zero modes then to contribute to the Yukawa couplings determined by the superpotential. As one insertion of each mass suffices to lift these modes the resulting term in the superpotential is naturally proportional to $m_1 m_2 m_3$. Apart from this the other factors in the instanton measure are essentially the same as in the $\mathcal{N}=4$ theory and in particular, the k-instanton contribution is again proportional to the Gauss-Bonnet integral over the relative moduli space of k BPS monopoles. As described below, the only further complication arises from the non-cancellation of the functional determinants which account for the contribution of non-zero modes.

To focus on the monopole contributions of winding number $n=0$, it is convenient to compare the exact superpotential with a semiclassical calculation in the three dimensional limit. Hence we take $\beta \to 0$ and $g^2 \to 0$ with the three dimensional gauge coupling $e^2 = 2\pi g^2/\beta$ held fixed. We also take the Wilson to zero; $\omega \to 0$, with $\phi_7 = \omega/\beta$ held fixed. To specify a classical vacuum of the $\mathcal{N}=1^*$ theory we return to case 1 defined above with the zero VEVs for the four dimensional scalar fields $\phi^a = 0$ for $a = 1, \ldots, 6$. In this limit the $n > 0$ terms in the expansion (10) vanish and the exact superpotential of [7] becomes,

$$\mathcal{W}(Z) = m_1 m_2 m_3 \, \frac{1}{\sinh^2(Z/2)} = \sum_{k=1}^{\infty} k \exp(-kZ)\,, \qquad (36)$$

where, at leading semiclassical order, $Z = S_{cl} - i\sigma$ with $S_{cl} = 8\pi^2\phi_7/e^2$ being the euclidean action of a single 3D instanton. In fact, we will show that this definition of Z in terms of ϕ_7 and σ is modified by perturbative effects. To see this we must compute the one-loop correction to the tree-level effective action given as (4) above which in three-dimensions becomes

$$S_B = \frac{e^2}{\pi(8\pi)^2} \int d^3x \left[\left(\frac{4\pi}{e^2} \partial_\mu \phi_7 \right)^2 + (\partial_\mu \sigma)^2 \right]. \qquad (37)$$

The correction can be expressed as a one-loop renormalization of the 3D coupling e^2 which has been calculated in the [27, appendix A],

$$\frac{2\pi}{e^2} \longrightarrow \frac{2\pi}{e^2} \left(1 - \frac{3}{S_{cl}} + \sum_{i=1}^{3} (S_i^2 + S_{cl}^2)^{-1/2} \right), \qquad (38)$$

where $S_i = 8\pi^2 |m_i|/e^2$.

One may deduce the one-loop modification in the definition of the field Z, by requiring the kinetic term of the effective theory to be manifestly $\mathcal{N}=1$ supersymmetric in terms of the complex superfield Z,

$$S_{\text{eff}} = \int d^3x g_{\bar{Z}Z} \, \partial_\mu \bar{Z} \partial^\mu Z\,, \qquad (39)$$

with $g_{\bar{Z}Z} = \partial_Z \partial_{\bar{Z}} \mathcal{K}(Z, \bar{Z})$ for some Kähler potential. One can show that the solution is given by the following one-loop definition,

$$Z = S_{cl} - 3\ln S_{cl} + \frac{1}{2} \ln \prod_{i=1}^{3} \frac{(S_i^2 + S_{cl}^2)^{1/2} + S_{cl}}{(S_i^2 + S_{cl}^2)^{1/2} - S_{cl}} + \ln \prod_{i=1}^{3} \frac{S_i}{2} - i\frac{\theta}{2\pi}\phi - i\sigma\,. \qquad (40)$$

The corresponding Kähler metric is

$$g_{\bar{Z}Z} = \frac{e^2}{\pi(8\pi)^2}\left(1 - \frac{3}{S_{cl}} + \sum_{i=1}^{3}(S_i^2 + S_{cl}^2)^{-1/2}\right)^{-1} . \tag{41}$$

Note that the addition of a constant term to Z in (40) does not change the effective action (39).

This superpotential yields a Yukawa coupling in the effective lagrangian of the form,

$$\mathcal{L}_{2f} = \left(\frac{2\pi}{e^2}\right)^3 (4\pi)^2\, \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}\, m_1 m_2 m_3 \sum_{k=1}^{\infty} k^3 \exp\left(-kZ\right), \tag{42}$$

where the fermions λ_α and $\bar{\lambda}_{\dot{\alpha}}$ are the low energy components of the Weyl fermions in the microscopic theory. Note that the kinetic term for the $\mathcal{N}=1$ superpartners of the complex scalar Z has a normalization which is different from that of the microscopic theory, hence one must take into account these normalization factors in the vertex (42). It is clear that all numbers of instantons contribute to a two fermion vertex, reflecting the fact that instantons are invariant under half the SUSY generators of the unbroken $\mathcal{N}=1$ supersymmetry.

In order to check this prediction we will calculate the large distance behaviour of the two fermion correlation function. correlation function;

$$\mathcal{G}^{(2)}(x_1, x_2)_{\alpha\,\beta} = \langle\, \lambda_\alpha(x_1)\, \lambda_\beta(x_2)\, \rangle \tag{43}$$

in the leading semiclassical approximation. The semiclassical approximation is valid when $\phi_7 \gg e^2$ so that $Z \gg 1$. The calculation is very similar to those appearing in [3] and [27], hence we will mostly emphasize the new features. The appropriate instanton measure for the three dimensional theory without mass terms has been worked out in detail in [3] and here we simply quote the relevant formulae. The measure for integration over the bosonic COM coordinates $\vec{R}$ and χ is,

$$\int d\bar{\mu}_B = \int \frac{d^3R}{(2\pi)^{3/2}}(g_{RR})^{3/2} \int_0^{2\pi/k} \frac{d\chi}{(2\pi)^{1/2}}(g_{\chi\chi})^{1/2}, \tag{44}$$

where the limit of integration on the χ-integral reflects the discrete Z_k symmetry. The overall normalization of the translational and charge rotation zero modes of a single monopole. The jacobian factors appearing in (44) are given explicitly as $g_{RR} = kS_{cl}$ and $g_{\chi\chi} = kS_{cl}/\phi_7^2$. These constants are related to the mass and moment of inertia of the k-monopole solution respectively.

As above, the eight exact zero modes occuring in the theory without mass terms can parametrized by four Grassmann spinor collective coordinates, ξ_α^A with $A = 1,2,3,4$. The corresponding contribution to the multi-instanton measure is,

$$\int d\bar{\mu}_F = \int \prod_{M=1}^{4} d^2\xi_M\, (k\mathcal{J}_\xi)^{-4}\,, \tag{45}$$

where the normalization constant $\mathcal{J}_\xi = 2S_{cl}$ was determined in [28]. Finally, the modes corresponding to the coordinates Y_q $q = 1, \ldots, d = 4(k-1)$ on the reduced moduli space $\tilde{\mathcal{M}}_k$ and their superpartners contribute an additional overall factor to the measure equal to $\mathcal{Z}_k^{\rm Rel}$ just as in [3], where $\mathcal{Z}_k^{\rm Rel}$ is equal to the Gauss-Bonnet integral over $\tilde{\mathcal{M}}_k$ defined in (30) above. In the light of our previous results we will assume this is equal to k.

Introducing non-zero masses m_i breaks $\mathcal{N}=4$ supersymmetry down to an $\mathcal{N}=1$ subalgebra. Of the four left-handed Weyl supercharges of the $\mathcal{N}=4$ theory, denoted Q^A_α with $A = 1,2,3,4$, only one remains unbroken. Without loss of generality we can choose this to be $Q_\alpha = Q^4_\alpha$ while we denote the broken supercharges $Q^i_\alpha = Q^A_\alpha$ for $i = A = 1,2,3$. We will use the same notation for the decomposition for the fermion fields λ^A_α and Grassmann collective coordinates ξ^A_α. The mass terms, when brought down from the action, saturate the integration over the collective coordinates $\xi_{\alpha\, i}$ with $i = 1,2,3$. The zero-mode part of the mass term,

$$S_{\rm mass} = \left(\frac{2\pi}{e^2}\right) \int d^3x \sum_{i=1}^{3} m_i {\rm Tr}\, \lambda_i^{cl\;\alpha} \lambda^{cl}_{\alpha\, i}\,, \tag{46}$$

produces a factor of $m_1 m_2 m_3$ when integrated over ξ_i. (Here $\lambda^{cl}_{\alpha\; M} = 8\pi k \xi^\beta_M S_F(\vec{r} - \vec{R})^\beta_\alpha$ with $S_F(\vec{r}) = \tau_i r^i / 4\pi |\vec{r}|^3$ being the three-dimensional fermion propagator, denotes the large-distance behaviour of the fermionic fields in the instanton background.) The two remaining zero modes of the gluino fields λ_α are not lifted and thus the remaining integration over ξ_4 is saturated saturated by the explicit fermionic insertions in (43).

Another important effect which appears after $\mathcal{N}=4$ supersymmetry is broken to an $\mathcal{N}=1$ subalgebra is that the determinants corresponding to non-zero eigenvalues of the fluctuation operators in the instanton background no longer cancel exactly between bosons and fermions. The residual factor is equal to,

$$\mathcal{R} = (2S_{cl})^{3k} \prod_{i=1}^{3} (S_i)^{-k} \left(\frac{(S_i^2 + S_{cl}^2)^{1/2} + S_{cl}}{(S_i^2 + S_{cl}^2)^{1/2} - S_{cl}}\right)^{-k/2}. \tag{47}$$

Putting these pieces together we find that,

$$\mathcal{G}^{(2)}(x_1, x_2)_{\alpha\beta} = \left(\frac{2\pi}{e^2}\right) m_1 m_2 m_3\, \mathcal{R} \exp(-kS_{cl} + ik\sigma)\, \mathcal{Z}_k^{\rm Rel} \times \\ \times \int d^3R \int d^2\xi_4\, \lambda^{cl}_\alpha(x_1) \lambda^{cl}_\beta(x_2)\,. \tag{48}$$

Note that the overall factor $\mathcal{R}\exp(-kS_{cl} + ik\sigma)$ is precisely equal to $\exp(-kZ)$ using the one-loop definition of Z given in (40) above. In other words the factor coming from the non-cancelation of determinants simply impliments the one-loop renormalization of the coupling. The resulting correlation function can then be written as,

$$\mathcal{G}^{(2)}(x_1,x_2)_{\alpha\beta} = \left(\frac{8\pi}{e}\right)^2 \pi\, m_1 m_2 m_3\, k^3\, \exp(-kZ) \int d^3R\, S_F(x-R)_\alpha^{\ \gamma} S_F(x_2-R)_{\beta\gamma}\,. \tag{49}$$

One may easily check that this coincides with the tree level contribution of the predicted vertex (42).

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